Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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Matrices of sizes 1×1, 2×2 and 3×3

Not a definition

Any SQUARE matrix $A \in \mathbb{K}_n^n$ over a field $\mathbb{K}$ is assigned a special scalar from $\mathbb{K}$. This scalar will be denoted by $|A|$ or $\det(A)$ and called *the determinant of* A.

Determinants for square matrices of sizes 1×1, 2×2 and 3×3:

$$\begin{vmatrix} a_{11} \end{vmatrix} = a_{11},$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32} \\ -a_{13} a_{22} a_{31} - a_{12} \cdot a_{21} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}. \end{array}$$

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Determinant: general definition

Definition

Determinant of a square matrix A of size $n \times n$, denoted by $\det(A)$ or $|A|$, is defined by

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \cdot a_{1\ \sigma(1)} \cdot a_{2\ \sigma(2)} \cdot \ldots \cdot a_{n\ \sigma(n)}$$

Recall

$$S_3 = \{123, 132, 321, 213, 312, 231\},$$
$$sgn(123) = sgn(231) = sgn(321) = 1,$$
$$sgn(132) = sgn(213) = sgn(321) = -1.$$

Laplace expansion

Let A be an $n \times n$ matrix. A matrix A_{ij} is obtained from A by deleting the i-th row and j-th column.

Laplace expansion

Let j be any number between 1 and n. Then:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij}).$$

expansion of det

The above formula is called the Laplace along the j-th column.

Laplace expansion

In particular, if $j = 1$ then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \cdot \det(A_{i1}).$$

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Laplace expansion

Let's calculate the determinant of the following matrix by expanding it along the first column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \det(A_{11}) - 4 \cdot \det(A_{21}) + 7 \cdot \det(A_{31}) =$$

$$1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 0$$

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Properties of determinants

Let A be a square matrix of size $n \times n$.

Fact

1. $\det(A) = \det(A^T)$,
2. $\det(A) = 0$ if A has a zero row (column) or two identical rows (columns),
3. $\det(A) = 0$ if and only if $r(A) < n$,
4. $\det(A) = -\det(B)$ if B is obtained from A by single row switching ($R_i \leftrightarrow R_j$),
5. $\det(A) = \det(B)$ if B is obtained from A by single row addition ($R_i + k \cdot R_j \to R_i$),
6. det(B) = k det(A)
   ~~$\det(A) = k \cdot \det(B)$~~ if B is obtained from A by the row scaling $kR_i \to R_i$.

The three latter properties are also true for elementary column operations.

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Properties of determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \overset{R_2 - 4\cdot R_1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix} \overset{R_3 - 7\cdot R_1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} =$$

$$2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{vmatrix} = 2 \cdot 0 = 0.$$

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Properties of determinants

Fact

$$\det(I_{n\times n}) = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \dots & \dots & \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1.$$

Fact

Let A and B be square matrices of size $n \times n$.

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

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Determinants and systems of linear equation

Theorem

If $AX = B$ is a system of n linear equations with n unknowns then $AX = B$ has a unique solution iff

$$\det(A) \neq 0.$$

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Solving SoLEs using determinants

Cramer's rule

Consider a system $AX = B$ of linear equations with n equations and n unknowns. In other words, with A a square matrix of size $n \times n$. Let $A_{|i}$ denote a matrix obtained from A by replacing its i-th column with the column B. If $\det(A) \neq 0$ then

$$
\begin{aligned}
x_1 &= \frac{\det(A_{|1})}{\det(A)},\\
x_2 &= \frac{\det(A_{|2})}{\det(A)},\\
&\ldots\\
x_n &= \frac{\det(A_{|n})}{\det(A)}.
\end{aligned}
$$

Solving SoLEs using determinants: example

Consider the system

$$x + y = 1,$$
$$x + 2y = 0.$$

Here, we have

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ \det(A) = 1.$$

Moreover,

$$A_{|1} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \det(A_{|1}) = 2,$$
$$A_{|2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \det(A_{|2}) = -1.$$

Hence, $x = 2$ and $y = -1$.

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Matrix inversion

Definition

Let A a square matrix of size $n \times n$. A matrix B of the same size as A is called *inverse* of A if

$$A \cdot B = I_{n\times n},$$
$$B \cdot A = I_{n\times n}.$$

If B is an inverse of A then it is unique.

The inverse of A is denoted by A^{-1}.

Example

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

It is easy to check that the following matrix in the inverse of A:

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Warning

NOT all matrices are invertible.

When is a matrix invertible?

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$. If A is invertible then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof (of the 2nd statement): Recall that $\det(I) = 1$ and $\det(A \cdot B) = \det(A) \cdot \det(B)$. If A is invertible then $A \cdot A^{-1} = I$. Hence,

$$1 = \det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}).$$

How to invert a matrix?

Fact

Let A be a square invertible matrix. Consider the $n \times 2n$ matrix

$$C = (A|I_{n \times n}).$$

Row reduce the matrix C to the following form

$$(I_{n \times n}|B).$$

Such a reduction is possible if and only if A is invertible. Then B obtained above is the inverse of A.

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right)$$

Inversion using determinants

Let A be an invertible square matrix.

Fact

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} (-1)^{1+1}|A_{11}|, & (-1)^{1+2}|A_{12}|, & \dots & (-1)^{1+n}|A_{1n}| \\ (-1)^{2+1}|A_{21}|, & (-1)^{2+2}|A_{22}|, & \dots & (-1)^{2+n}|A_{2,n}| \\ \dots & \dots & \dots & \dots \\ (-1)^{n+1}|A_{n,1}|, & (-1)^{n+2}|A_{n,2}|, & \dots & (-1)^{n+n}|A_{n,n}| \end{pmatrix}^T$$

where $A_{k,m}$ denotes a matrix obtained from A by deleting k-th row and m-th column.

Inversion using determinants

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} |(2)| & -1\cdot|(2)| \\ -1\cdot|(0)| & |(1)| \end{pmatrix}^{T} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$$

Inverting 2×2 matrices

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Assume that $\det(A) = ad - bc \neq 0$. Then

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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