Transition Maths and Algebra with Geometry

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Lecture Notes Electrical and Computer Engineering









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2 Determinants and systems of linear equations

3 Matrix inversion









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Matrices of sizes 1×1 , 2×2 and 3×3

Not a definition

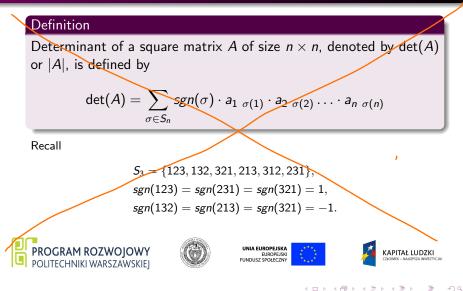
Any SQUARE matrix $A \in \mathbb{K}_n^n$ over a field \mathbb{K} is assigned a special scalar from \mathbb{K} . This scalar will be denoted by |A| or det(A) and called *the determinant of A*.

Determinants for square matrices of sizes 1×1 , 2×2 and 3×3 :

$$\begin{aligned} |a_{11}| &= a_{11}, \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11} \cdot a_{22} - a_{12} \cdot a_{21}, \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{array}{c} a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32} \\ -a_{13}a_{22}a_{31} - a_{12} \cdot a_{21} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}. \end{array}$$

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Determinant: general definition



Laplace expansion

Let A be an $n \times n$ matrix. A matrix A_{ij} is obtained from A by deleting the *i*-th row and *j*-th column.

Laplace expansion

Let j be any number between 1 and n. Then:

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \cdot \det(A_{ij}).$$

Expansion of det The above formula is called the Laplace along the j-th column.



Laplace expansion

In particular, if j = 1 then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \cdot \det(A_{i1}).$$







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Laplace expansion

Let's calculate the determinant of the following matrix by expanding it along the first column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \det(A_{11}) - 4 \cdot \det(A_{21}) + 7 \cdot \det(A_{31}) = \\ 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 0$$



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Properties of determinants

Let A be a square matrix of size $n \times n$.

Fact

2 det(A) = 0 if A has a zero row (column) or two identical rows (columns),

3 det
$$(A) = 0$$
 if and only if $r(A) < n$,

- det(A) = -det(B) if B is obtained from A by single row switching $(R_i \leftrightarrow R_j)$,
- det(A) = det(B) if B is obtained from A by single row addition $(R_i + k \cdot R_i \rightarrow R_i),$ det(B) = k det(A) if B is obtained from A by the row cooling b
- det(A) = k det(B) if B is obtained from A by the row scaling $kR_i \to R_i$.

The three latter properties are also true for elementary column operations.









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Properties of determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \stackrel{R_2 - 4 \cdot R_1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix} \stackrel{R_3 - 7 \cdot R_1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = 2 \cdot 0 = 0.$$









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Properties of determinants

$\det(I_{n \times n}) = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \dots & \dots & \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1.$

Fact

Fact

Let A and B be square matrices of size $n \times n$.

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

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Determinants and systems of linear equations

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Determinants and systems of linear equation

Theorem

If AX = B is a system of *n* linear equations with *n* unknowns then AX = B has a unique solution iff

$$det(A) \neq 0.$$







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Solving SoLEs using determinants

Cramer's rule

Consider a system AX = B of linear equations with n equations and n unknowns. In other words, with A a square matrix of size $n \times n$. Let $A_{|i|}$ denote a matrix obtained from A by replacing its *i*-th column with the column B. If det $(A) \neq 0$ then

$$egin{aligned} &x_1=rac{\det(A_{|1})}{\det(A)},\ &x_2=rac{\det(A_{|2})}{\det(A)}, \end{aligned}$$

$$x_n = \frac{\det(A_{|n})}{\det(A)}.$$

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Solving SoLEs using determinants: example

Consider the system

$$\begin{aligned} x+y &= 1, \\ x+2y &= 0. \end{aligned}$$

Here, we have

$$A = \left(egin{array}{cc} 1 & 1 \ 1 & 2 \end{array}
ight), \; {
m det}(A) = 1.$$

Moreover,

$$egin{aligned} &A_{|1}=\left(egin{aligned} 1 & 1 \ 0 & 2 \end{array}
ight), \det(A_{|1})=2, \ &A_{|2}=\left(egin{aligned} 1 & 1 \ 1 & 0 \end{array}
ight), \det(A_{|2})=-1. \end{aligned}$$

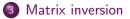


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2 Determinants and systems of linear equations











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Matrix inversion

Definition

Let A a square matrix of size $n \times n$. A matrix B of the same size as A is called *inverse* of A if

$$A \cdot B = I_{n \times n},$$
$$B \cdot A = I_{n \times n}.$$

If B is an inverse of A then it is unique.

The inverse of A is denoted by A^{-1} .







Example

Let

$$A = \left(\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}\right)$$

It is easy to check that the following matrix in the inverse of A:

$$A^{-1} = \left(\begin{array}{cc} 1 & 0\\ 0 & \frac{1}{2} \end{array}\right)$$



When is a matrix invertible?

Theorem

A square matrix A is invertible if and only if $det(A) \neq 0$. If A is invertible then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof (of the 2nd statement): Recall that det(I) = 1 and $det(A \cdot B) = det(A) \cdot det(B)$. If A is invertible then $A \cdot A^{-1} = I$. Hence,

$$1 = \det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}).$$





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How to invert a matrix?

Fact

Let A be a square invertible matrix. Consider the $n \times 2n$ matrix

 $C = (A|I_{n \times n}).$

Row reduce the matrix C to the following form

 $(I_{n\times n}|B).$

Such a reduction is possible if and only if A is invertible. Then B obtained above is the inverse of A.



Inversion using determinants

Let A be an invertible square matrix.

Fact

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} (-1)^{1+1} |A_{11}|, & (-1)^{1+2} |A_{12}|, & \dots & (-1)^{1+n} |A_{1n}| \\ (-1)^{2+1} |A_{21}|, & (-1)^{2+2} |A_{22}|, & \dots & (-1)^{2+n} |A_{2,n}| \\ \dots & \dots & \dots & \dots \\ (-1)^{n+1} |A_{n,1}|, & (-1)^{n+2} |A_{n,2}|, & \dots & (-1)^{n+n} |A_{n,n}| \end{pmatrix}^{T}$$

where $A_{k,m}$ denotes a matrix obtained from A by deleting k-th row and m-th column.



Inversion using determinants

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} |(2)| & -1 \cdot |(2)| \\ -1 \cdot |(0)| & |(1)| \end{pmatrix}^{T} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$$







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Inverting 2×2 matrices

Let

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

Assume that $det(A) = ad - bc \neq 0$. Then

$$A^{-1} = rac{1}{ad-bc} \cdot \left(egin{array}{cc} d & -b \ -c & a \end{array}
ight)$$







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